Perturbative approach to open circuit QED systems

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Motivation – Quantum Simulators

Jaynes-Cummings lattice
- Closed systems: Superfluid (SF) to Mott Insulator (MI) phase transition
- Circuit QED systems: zero chemical potential for photon, external drive, photon leakage and qubit relaxation.

=> Open systems
Outline

• Perturbation Theory (PT) for Markovian open quantum systems

• PT vs exact result: good agreement
Master equation

- Master equation:
  \[
  \frac{d\rho}{dt} = \mathbb{L}\rho = -i [H, \rho] + \mathbb{D}\rho
  \]

- Eigenvalues problem:
  \[
  \mathbb{L}u_\mu = \lambda_\mu u_\mu, \quad \mathbb{L}^\dagger w_\mu = \lambda_\mu^* w_\mu
  \]
  \[\therefore \mathbb{L} \text{ may not be diagonalizable}\]
  \[\therefore u_\mu \text{ and } w_\mu \text{ may not be complete}\]

- Steady state \(\rho_s\):
  \[
  \frac{d\rho_s}{dt} = \mathbb{L}\rho_s = 0
  \]
  \[\Rightarrow \rho_s : \text{right eigenstate with eigenvalue zero}\]
Perturbation Theory (PT)

- Difficult to obtain the eigenvalues and eigenstates e.g. 16 spins
  \( L : 4^{16} \times 4^{16} \approx 4 \text{ billion} \times 4 \text{ billion!} \)

- Approximation: PT (analogous to closed-system PT) separate the Liouville superoperator

\[ L = L_0 + \alpha L_1 \]

Unperturbed

Small parameter

Perturbation
Previous work on open-system PT


- Series expansion of the steady state

\[ \rho_S = \sum_{j \geq 0} \alpha^j \rho_s^{(j)} \]

- Cannot apply to other eigenstates of L

- Positivity of \( \rho_S \) is not mentioned
Density-matrix PT

- The zeroth-order eigenvalue problem can be solved.

\[ \mathbb{L}_0 u^{(0)}_\mu = \lambda^{(0)}_\mu u^{(0)}_\mu \quad \mathbb{L}^\dagger_0 w^{(0)}_\mu = (\lambda^{(0)}_\mu)^* w^{(0)}_\mu \]

- Series expansion

\[ \mathbb{L} = \mathbb{L}_0 + \alpha \mathbb{L}_1 \quad \lambda_\mu = \sum_{j=0}^{\infty} \alpha^j \lambda^{(j)}_\mu \quad u_\mu = \sum_{j=0}^{\infty} \alpha^j u^{(j)}_\mu \]

Substitution

\[ \mathbb{L} u_\mu = \lambda_\mu u_\mu \]
Density-matrix PT

non-invertible and not necessarily diagonalizable

• Group terms of the same order of $\alpha$

\[
\left( \mathbb{L}_0 - \lambda^{(0)}_{\mu} \right) u^{(j)}_{\mu} = -\mathbb{L}_1 u^{(j-1)}_{\mu} + \Delta^{(j)}_{\mu}
\]

Solve

\[
\lambda^{(j)}_{\mu} = \left\langle w^{(0)}_{\mu}, \mathbb{L}_1 u^{(j-1)}_{\mu} \right\rangle - \sum_{k=1}^{j-1} \lambda^{(k)}_{\mu} \left\langle w^{(0)}_{\mu}, u^{(j-k)}_{\mu} \right\rangle
\]

\[
u^{(j)}_{\mu} = \left( \mathbb{L}_0 - \lambda^{(0)}_{\mu} \right)^{j-1} \left( -\mathbb{L}_1 u^{(j-1)}_{\mu} + \Delta^{(j)}_{\mu} \right)
\]

inner product

$\langle x, y \rangle \equiv \text{Tr} \left[ x^\dagger y \right]$
Steady state: issue of positivity

- Steady state
  \[ \rho_s = \sum_{j=0}^{\infty} \alpha^j \rho_s^{(j)} \]

- Truncation to \( M \)-th order
  \[ \rho_{s;M}^a = \sum_{j=0}^{M} \alpha^j \rho_s^{(j)} \]

\( \because \) probability is non-negative, positive-semidefinite

not necessary to be positive
Since density matrix is positive-semidefinite, 
\[ \rho = \zeta \zeta^\dagger \]  
\( \zeta \) amplitude matrix

If \( \rho_s \) is positive-definite, \( \zeta_s \) is uniquely defined by Cholesky decomposition and has a series expansion  
\[ \zeta_s = \sum_{j=0}^{\infty} \alpha^j \zeta^{(j)}_s \]

\[ \rho_{s;M}^{\text{a};M} = \left( \sum_{j=0}^{M} \alpha^j \zeta^{(j)}_s \right) \left( \sum_{j=0}^{M} \alpha^j \zeta^{(j)}_s \right)^\dagger \] is always positive

Determination of each order:

\[ \zeta^{(0)}_s \left( \zeta^{(0)}_s \right)^\dagger = \rho_s^{(0)} \]  
(Cholesky decomposition)

\[ \mathbb{Z}_0 \zeta^{(j)}_s = \rho_s^{(j)} - \sum_{k=1}^{j-1} \zeta^{(k)}_s \left( \zeta^{(j-k)}_s \right)^\dagger \]  
(Recursive relation)

where  
\[ \mathbb{Z}_0 \bullet = \zeta^{(0)}_s (\bullet)^\dagger + \bullet (\zeta^{(0)}_s)^\dagger \]
In frame co-rotating with the drive:

\[ H = \sum_n \left[ \delta \omega \sigma_n^+ \sigma_n^- + \epsilon (\sigma_n^+ + \sigma_n^-) \right] + t \sum_n (\sigma_n^+ \sigma_{n+1}^- + \text{h.c.}) \]

\[ \mathbb{L} \rho = -i[H, \rho] + \gamma \sum_n \mathbb{D} [\sigma_n^-] \rho \]
Second-order PT

Drive strength: \( \frac{\varepsilon}{\gamma} = 0.8 \)

Spin-spin coupling strength: \( \frac{t}{\gamma} = 0.4 \)
Summary

• Density matrix PT
  – perturbative correction to eigenvalues and eigenstates of \( L \)
    (e.g. steady state)

• Amplitude matrix PT
  – tackles the issue of positivity

• Two schemes provide similar accuracy for expectation values

• Outlook: lattice systems
  – systematic hierarchy of coupled clusters of sites